Sums of generalized harmonic series for kids from five to fifteen

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Abstract

We reexamine remarkable connection, first discovered by Beukers, Kolk and Calabi, between $\zeta(2n)$, the value of the Riemann zeta-function at even positive integer, and the volume of some $2n$-dimensional polytope. It can be shown that this volume equals to the trace of some compact self-adjoint operator. We provide an explicit expression for the kernel of this operator in terms of Euler polynomials. This explicit expression makes it easy to calculate the volume of the polytope and hence $\zeta(2n)$. In the case of odd positive integers, the expression for the kernel enables to rediscover an integral representation for $\zeta(2n+1)$, obtained originally by different method by Cvijović and Klinowski. Finally, we indicate that the origin of the Beukers-Kolk-Calabi's miraculous change of variables in the multidimensional integral, which is at the heart of all of this business, can be traced down to the amoeba associated with the certain Laurent polynomial.

1 Introduction

In a nice little book [1] Vladimir Arnold has collected 77 mathematical problems for Kids from 5 to 15 to stimulate the development of a culture of critical thinking in pupils. Problem 51 in this book asks to calculate the sum of inverse squares of the positive integers and prove the Euler's celebrated formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1}$$

Well, there are many ways to do this (see, for example, [2, 3, 4, 5, 6, 7] and references therein), some maybe even accessible for kids under fifteen. However, in this note we concentrate on the approach of Beukers, Kolk and Calabi [8], further elaborated by Elkies in [9]. This approach incorporates pleasant features which all the kids (and even some adults) adore: simplicity, magic and the depth that allows to go beyond the particular case (1). The simplicity, however, is not everywhere explicit in [8] and [9], while the magic longs for explanation after the first admiration fades away. Below we will try to enhance the simplicity of the approach and somewhat uncover the secret of magic.

The paper is organized as follows. In the first two sections we reconsider the evaluation of $\zeta(2)$ and $\zeta(3)$ in order technical details of the general case not to obscure the simple underlying ideas. Then we elaborate the general case and provide the main result of this work, the formula for the kernel which allows to simplify considerably the evaluation of $\zeta(2n)$ from [8, 9] and re-derive Cvijović and Klinowski's integral representation [10] for $\zeta(2n+1)$. Finally, we ponder over the mysterious relations between the sums of generalized harmonic series and amoebas, first indicated by Passare in [7]. This relation enables to somewhat uncover the origin of the Beukers-Kolk-Calabi's highly non-trivial change of variables.

2 Evaluation of $\zeta(2)$

Recall the definition of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2}$$

The sum (1) is just $\zeta(2)$ which we will now evaluate following the method of Beukers, Kolk and Calabi [8]. Our starting point will be the dilogarithm function

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}. \tag{3}$$

Clearly, $Li_2(0) = 0$ and $Li_2(1) = \zeta(2)$. Differentiating (3), we get

$$x\,\frac{d}{dx}Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln{(1-x)},$$

and, therefore,

$$\zeta(2) = Li_2(1) = -\int\limits_0^1 \frac{\ln(1-x)}{x} dx = \iint\limits_{\square} \frac{dx\,dy}{1-xy}, \tag{4}$$

where $\square = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$ is the unit square. Let us note

$$\iint\limits_{\square} \frac{dx\,dy}{1-xy} + \iint\limits_{\square} \frac{dx\,dy}{1+xy} = 2\iint\limits_{\square} \frac{dx\,dy}{1-x^2y^2}, \tag{5}$$

and

$$\iint\limits_{\square} \frac{dx\,dy}{1-xy} - \iint\limits_{\square} \frac{dx\,dy}{1+xy} = \frac{1}{2}\iint\limits_{\square} \frac{dx\,dy}{1-xy}, \tag{6}$$

where the last equation follows from

$$\iint\limits_{\square} \frac{2xy}{1-x^2y^2} dx\,dy = \frac{1}{2}\iint\limits_{\square} \frac{dx^2\,dy^2}{1-x^2y^2} = \frac{1}{2}\iint\limits_{\square} \frac{dx\,dy}{1-xy}.$$

It follows from equations (5) and (6) that

$$\zeta(2) = \frac{4}{3}\iint\limits_{\square} \frac{dx\,dy}{1-x^2y^2}. \tag{7}$$

Now let us make the Beukers-Kolk-Calabi's magic change of variables in this two-dimensional integral [8]

$$x = \frac{\sin u}{\cos v}, \quad y = \frac{\sin v}{\cos u}, \tag{8}$$

with the Jacobi determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\cos u}{\cos v} & \frac{\sin v\,\sin u}{\cos^2 u} \\ \frac{\sin u\,\sin v}{\cos^2 v} & \frac{\cos v}{\cos u} \end{vmatrix} = 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 v \cos^2 u} = 1 - x^2y^2.$$

Then miraculously

$$\zeta(2) = \frac{4}{3}\iint\limits_{\Delta} du\,dv = \frac{4}{3}\,Area(\Delta), \tag{9}$$

where Δ is the image of the unit square $\square$ under the transformation $(x, y) \to (u, v)$. It is easy to realize that Δ is the isosceles right triangle $\Delta = \{(u, v) : u \geq 0, v \geq 0, u + v \leq \pi/2\}$ and, therefore,

$$\zeta(2) = \frac{4}{3}\frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{6}. \tag{10}$$

"Beautiful – even more so, as the same method of proof extends to the computation of $\zeta(2k)$ in terms of a $2k$-dimensional integral, for all $k \geq 1$" [11]. However, before considering the general case, we check whether the trick works for $\zeta(3)$.

3 Evaluation of $\zeta(3)$

In the case of $\zeta(3)$, we begin with trilogarithm

$$Li_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}, \tag{11}$$

and using

$$x\,\frac{d}{dx}Li_3(x) = Li_2(x) = -\int_0^x \frac{\ln(1-y)}{y}dy,$$

we get

$$\zeta(3) = Li_3(1) = -\int_0^1 \frac{dx}{x}\int_0^x \frac{\ln(1-y)}{y}dy. \tag{12}$$

But

$$-\frac{1}{x}\int_0^x \frac{\ln(1-y)}{y}dy = -\int_0^1 \frac{\ln(1-xz)}{xz}dz = \int_0^1 dz\int_0^1 \frac{dy}{1-xyz},$$

and finally

$$\zeta(3) = Li_3(1) = \iiint_{\square_3} \frac{dx\,dy\,dz}{1-xyz}, \tag{13}$$

where $\square_3 = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ is the unit cube. By the similar trick as before, we can transform (13) into the integral

$$\zeta(3) = \frac{8}{7} \iiint\limits_{\square_3} \frac{dx\,dy\,dz}{1 - x^2y^2z^2}, \tag{14}$$

and here the analogy with the previous case ends, unfortunately, because the generalization of the Beukers-Kolk-Calabi change of variables does not lead in this case to the simple integral. However, it is interesting to note that the hyperbolic version of this change of variables

$$x = \frac{\sinh u}{\cosh v}, \quad y = \frac{\sinh v}{\cosh w}, \quad z = \frac{\sinh w}{\cosh u} \tag{15}$$

does indeed produce an interesting result

$$\zeta(3) = \frac{8}{7} \iiint\limits_{U_3} du\,dv\,dw = \frac{8}{7}\,\mathrm{Vol}(U_3), \tag{16}$$

where U_3 is a complicated 3-dimensional shape defined by the inequalities

$$u \ge 0, \;\; v \ge 0, \;\; w \ge 0, \;\; \sinh u \le \cosh v, \;\; \sinh v \le \cosh w, \;\; \sinh w \le \cosh u.$$

Unfortunately, unlike the previous case, there is no obvious simple way to calculate the volume of U_3.

However, there is a second way to convert the integral (12) for $\zeta(3)$ in which the Beukers-Kolk-Calabi change of variables still plays a helpful role. We begin with the identity

$$\zeta(3) = -\int\limits_0^1 \frac{dx}{x} \int\limits_0^x \frac{\ln(1-y)}{y} dy = -\iint\limits_D \frac{\ln(1-y)}{xy}\,dx\,dy, \tag{17}$$

where the domain of the 2-dimensional integration is the triangle $D = \{(x,y) : x \ge 0, y \ge 0, y \le x\}$. Interchanging the order of x and y integrations in the evaluation of the 2-dimensional integral (17), we get

$$\zeta(3) = -\int\limits_0^1 \frac{\ln(1-y)}{y}\,dy \int\limits_y^1 \frac{dx}{x},$$

which can be transformed further as follows

$$\zeta(3) = \int\limits_0^1 \frac{\ln(1-y)\ln y}{y}\, dy = -\int\limits_0^1 \ln y\, dy \int\limits_0^1 \frac{dx}{1-xy} = -\iint\limits_{\square} \frac{\ln y}{1-xy}\, dx\, dy,$$

or in a more symmetrical form

$$\zeta(3) = -\frac{1}{2} \iint\limits_{\square} \frac{\ln(xy)}{1-xy}\, dx\, dy. \tag{18}$$

Note that

$$\iint\limits_{\square} \frac{2xy\, \ln(xy)}{1-x^2y^2}\, dx\, dy = \frac{1}{4} \iint\limits_{\square} \frac{\ln(x^2y^2)}{1-x^2y^2}\, dx^2\, dy^2 = \frac{1}{4} \iint\limits_{\square} \frac{\ln(xy)}{1-xy}\, dx\, dy.$$

Therefore, we can modify (5) and (6) accordingly and using them transform (18) into

$$\zeta(3) = -\frac{4}{7} \iint\limits_{\square} \frac{\ln(xy)}{1-x^2y^2}\, dx\, dy. \tag{19}$$

At this point we can use Beukers-Kolk-Calabi change of variables (8) in (19) and as a result we get

$$\zeta(3) = -\frac{4}{7} \iint\limits_{\Delta} \ln(\tan u\, \tan v)\, du\, dv = -\frac{8}{7} \iint\limits_{\Delta} \ln(\tan u)\, du\, dv. \tag{20}$$

But this equation indicates that

$$\zeta(3) = -\frac{8}{7} \int\limits_0^{\pi/2} du\, \ln(\tan u) \int\limits_0^{\pi/2-u} dv = -\frac{8}{7} \int\limits_0^{\pi/2} \left(\frac{\pi}{2} - u\right) \ln(\tan u)\, du,$$

which after substitution $x = \frac{\pi}{2} - u$ becomes

$$\zeta(3) = -\frac{8}{7} \int\limits_0^{\pi/2} x \ln(\cot x)\, dx = \frac{8}{7} \int\limits_0^{\pi/2} x \ln(\tan x)\, dx. \tag{21}$$

But

$$\int_0^{\pi/2} \ln(\tan x)\,dx = -\int_{\pi/2}^{0} \ln(\cot u)\,du = -\int_0^{\pi/2} \ln(\tan u)\,du = 0,$$

which enables to rewrite (21) as follows

$$\zeta(3) = \frac{8}{7}\int_0^{\pi/2} \left(x - \frac{\pi}{4}\right) \ln(\tan x)\,dx = \frac{8}{7}\int_0^{\pi/2} \ln(\tan x)\,\frac{d}{dx}\left(\frac{x^2}{2} - \frac{\pi}{4}x\right)\,dx,$$

and after integration by parts and rescaling $x \to x/2$ we end with

$$\zeta(3) = \frac{1}{7}\int_0^{\pi} \frac{x(\pi - x)}{\sin x}\,dx. \tag{22}$$

This is certainly an interesting result. Note that until quite recently very few definite integrals of this kind, involving cosecant or secant functions, were known and present in standard tables of integrals [12, 13, 14]. In fact (22) is a special case of the more general result [10] which we are going now to establish.

4 The general case of $\zeta(2n)$

The evaluation of $\zeta(2)$ can be straightforwardly generalized. The polylogarithm function

$$Li_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \tag{23}$$

obeys

$$x\,\frac{d}{dx} Li_s(x) = Li_{s-1}(x),$$

and hence

$$Li_s(x) = \int_0^x \frac{Li_{s-1}(y)}{y}\,dy. \tag{24}$$

Repeated application of this identity allows to write

$$\zeta(n) = Li_n(1) = \int\limits_0^1 \frac{dx_1}{x_1} \int\limits_0^{x_1} \frac{dx_2}{x_2} \dots \int\limits_0^{x_{n-2}} \frac{dx_{n-1}}{x_{n-1}} \left[-\ln\left(1 - x_{n-1}\right)\right]. \tag{25}$$

After rescaling

$$x_1 = y_1,\;\; x_2 = x_1 y_2,\;\; x_3 = x_2 y_3, \dots, x_{n-1} = x_{n-2} y_{n-1} = y_1 y_2 \cdots y_{n-1},$$

and using

$$\int\limits_0^1 \frac{dy_n}{1 - y_1 y_2 \cdots y_n} = -\frac{1}{y_1 y_2 \cdots y_{n-1}} \ln\left(1 - y_1 y_2 \cdots y_{n-1}\right),$$

we get

$$\zeta(n) = \int\cdots\int\limits_{\square_n} \frac{dy_1\, dy_2 \cdots dy_n}{1 - y_1 y_2 \cdots y_n}, \tag{26}$$

where $\square_n$ is n-dimensional unit hypercube. The analogs of (5) and (6) are

$$\int\cdots\int\limits_{\square_n} \frac{dx_1 \cdots dx_n}{1 - x_1 \cdots x_n} + \int\cdots\int\limits_{\square_n} \frac{dx_1 \cdots dx_n}{1 + x_1 \cdots x_n} = 2 \int\cdots\int\limits_{\square_n} \frac{dx_1 \cdots dx_n}{1 - x_1^2 \cdots x_n^2}$$

and

$$\int\cdots\int\limits_{\square_n} \frac{dx_1 \cdots dx_n}{1 - x_1 \cdots x_n} - \int\cdots\int\limits_{\square_n} \frac{dx_1 \cdots dx_n}{1 + x_1 \cdots x_n} = \frac{1}{2^{n-1}} \int\cdots\int\limits_{\square_n} \frac{dx_1 \cdots dx_n}{1 - x_1 \cdots x_n},$$

from which it follows that (26) is equivalent to

$$\zeta(n) = \frac{2^n - 1}{2^n} \int\cdots\int\limits_{\square_n} \frac{dx_1 \cdots dx_n}{1 - x_1^2 \cdots x_n^2}. \tag{27}$$

If we now make a change of variables that generalizes (8), namely

$$x_1 = \frac{\sin u_1}{\cos u_2},\;\; x_2 = \frac{\sin u_2}{\cos u_3}, \dots,\;\; x_{n-1} = \frac{\sin u_{n-1}}{\cos u_n},\;\; x_n = \frac{\sin u_n}{\cos u_1}. \tag{28}$$

we, in general, encounter a problem because the Jacobian of (28) is [8, 9]

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(u_1,\ldots,u_n)} = 1-(-1)^n\, x_1^2x_2^2\cdots x_n^2,$$

and, therefore, only for even n we will get a “simple” integral. For the hyperbolic version of (28),

$$x_1=\frac{\sinh v_1}{\cosh v_2},\quad x_2=\frac{\sinh v_2}{\cosh v_3},\ldots,x_{n-1}=\frac{\sinh v_{n-1}}{\cosh v_n},\quad x_n=\frac{\sinh v_n}{\cosh v_1}, \tag{29}$$

the Jacobian has a “right” form

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(v_1,\ldots,v_n)} = 1-x_1^2x_2^2\cdots x_n^2,$$

and we get

$$\zeta(n)=\frac{2^n}{2^n-1}\int\limits_{U_n}\cdots\int dv_1\cdots dv_n=\frac{2^n}{2^n-1}\,\mathrm{Vol}_n(U_n). \tag{30}$$

However, the figure U_n has a complicated shape and it is not altogether clear how to calculate its n-dimensional volume $\mathrm{Vol}_n(U_n)$. Therefore, for a moment, we concentrate on the even values of n for which (28) works perfectly well and leads to

$$\zeta(2n)=\frac{2^{2n}}{2^{2n}-1}\int\limits_{\Delta_{2n}}\cdots\int du_1\cdots du_n=\frac{2^{2n}}{2^{2n}-1}\,\mathrm{Vol}_{2n}(\Delta_{2n}), \tag{31}$$

where Δ_n is a n-dimensional polytope defined through the inequalities

$$\Delta_n=\left\{(u_1,\ldots,u_n): u_i\ge 0,\ u_i+u_{i+1}\le\frac{\pi}{2}\right\}. \tag{32}$$

It is assumed in (32) that u_i are indexed cyclically (mod n) and therefore $u_{n+1}=u_1$.

There exists an elegant method due to Elkies [9] how to calculate the n-volume of Δ_n (earlier calculations of this type can be found in [15]). Obviously

$$\mathrm{Vol}_n(\Delta_n)=\left(\frac{\pi}{2}\right)^n\,\mathrm{Vol}_n(\delta_n), \tag{33}$$

where $\mathrm{Vol}_n(\delta_n)$ is the n-dimensional volume of the rescaled polytope

$$\delta_n = \{(u_1, \dots, u_n) : u_i \geq 0,\ u_i + u_{i+1} \leq 1\}\,. \tag{34}$$

If we introduce the characteristic function $K_1(u, v)$ of the isosceles right triangle $\{(u, v) : u, v \geq 0,\ u + v \leq 1\}$ that is 1 inside the triangle and 0 outside of it, then [9]

$$\mathrm{Vol}_n(\delta_n) = \int_0^1 \dots \int_0^1 \prod_{i=1}^{n} K_1(u_i, u_{i+1})\, du_1 \dots du_n = \int_0^1 du_1 \int_0^1 du_2\, K_1(u_1, u_2) \dots$$

$$\int_0^1 du_{n-1}\, K_1(u_{n-2}, u_{n-1}) \int_0^1 du_n\, K_1(u_{n-1}, u_n)\, K_1(u_n, u_1). \tag{35}$$

Let us note that $K_1(u, v)$ can be interpreted [9] as the kernel of the linear operator $\hat{T}$ on the Hilbert space $L^2(0, 1)$, defined as follows

$$(\hat{T}f)(u) = \int_0^1 K_1(u, v) f(v)\, dv = \int_0^{1-u} f(v)\, dv. \tag{36}$$

Then (35) shows that $\mathrm{Vol}_n(\delta_n)$ equals just to the trace of the operator $\hat{T}^n$:

$$\mathrm{Vol}_n(\delta_n) = \int_0^1 K_n(u_1, u_1)\, du_1, \tag{37}$$

whose kernel $K_n(u, v)$ obeys the recurrence relation

$$K_n(u, v) = \int_0^1 K_1(u, u_1)\, K_{n-1}(u_1, v)\, du_1. \tag{38}$$

Surprisingly, we can find a simple enough solution of this recurrence relation. Namely,

$$\begin{aligned} K_{2n}(u, v) = (-1)^n \frac{2^{2n-2}}{(2n-1)!} \times \\ \left\{ \left[E_{2n-1}\left(\frac{u+v}{2}\right) + E_{2n-1}\left(\frac{u-v}{2}\right) \right] \theta(u - v) + \right. \\ \left. \left[E_{2n-1}\left(\frac{u+v}{2}\right) + E_{2n-1}\left(\frac{v-u}{2}\right) \right] \theta(v - u) \right\}, \end{aligned} \tag{39}$$

and

$$K_{2n+1}(u,v) = (-1)^n \frac{2^{2n-1}}{(2n)!} \times$$
$$\left\{ \left[E_{2n}\left(\frac{1-u+v}{2}\right) + E_{2n}\left(\frac{1-u-v}{2}\right) \right] \theta(1-u-v) + \right.$$
$$\left. \left[E_{2n}\left(\frac{1-u+v}{2}\right) - E_{2n}\left(\frac{u+v-1}{2}\right) \right] \theta(u+v-1) \right\}. \qquad (40)$$

In these formulas $E_n(x)$ are the Euler polynomials [16] and $\theta(x)$ is the Heaviside step function

$$\theta(x) = \begin{cases} 1, & if \;\; x > 0, \\ \frac{1}{2}, & if \;\; x = 0, \\ 0, & if \;\; x < 0. \end{cases}$$

After they are guessed, it is quite straightforward to prove (39) and (40) by induction using the recurrence relation (38) and the following properties of the Euler polynomials

$$\frac{d}{dx} E_n(x) = n E_{n-1}(x), \quad E_n(1-x) = (-1)^n E_n(x). \qquad (41)$$

In particular, after rather lengthy but straightforward integration we get

$$\int\limits_0^{1-u} K_{2n+1}(u_1, v)\, du_1 = K_{2n+2}(u,v) - X,$$

where

$$X = (-1)^{n+1} \frac{2^{2n}}{(2n+1)!} \left[E_{2n+1}\left(\frac{1+v}{2}\right) + E_{2n+1}\left(\frac{1-v}{2}\right) \right].$$

But

$$\frac{1-v}{2} = 1 - \frac{1+v}{2}$$

and the second identity of (41) then implies that $X = 0$.

Therefore the only relevant question is how (39) and (40) were guessed. Maybe the best way to explain the "method" used is to refer to the problem

13 from the aforementioned book [1]. To demonstrate the cardinal difference between the ways problems are posed and solved by physicists and by mathematicians, Arnold provides the following problem for children:

"On a bookshelf there are two volumes of Pushkin 's poetry. The thickness of the pages of each volume is 2 cm and that of each cover 2 mm. A worm holes through from the first page of the first volume to the last page of the second, along the normal direction to the pages. What distance did it cover?"

Usually kids have no problems to find the unexpected correct answer, 4 mm, in contrast to adults. For example, the editors of the highly respectable physics journal initially corrected the text of the problem itself into: "from the last page of first volume to the first page of the second" to "match" the answer given by Arnold [1, 17]. The secret of kids lies in the experimental method used by them: they simple go to the shelf and see how the first page of the first volume and the last page of the second are situated with respect to each other.

The method that led to (39) and (40) was exactly of this kind: we simply calculated a number of explicit expressions for $K_n(u,v)$ using (38) and tried to locate regularities in this expressions.

Having (39) at our disposal, it is easy to calculate the integral in (37). Namely, because

$$K_{2n}(u,u) = (-1)^n \, \frac{2^{2n-2}}{(2n-1)!} \left[E_{2n-1}(u) + E_{2n-1}(0) \right], \tag{42}$$

and

$$E_{2n-1}(u) = \frac{1}{2n} \frac{d}{du} E_{2n}(u), \tag{43}$$

we get

$$\mathrm{Vol}_{2n}(\delta_{2n}) = \int\limits_0^1 K_n(u,u)\, du = (-1)^n \, \frac{2^{2n-2}}{(2n-1)!} E_{2n-1}(0), \tag{44}$$

(note that $E_{2n}(0) = E_{2n}(1) = 0$.) But $E_{2n-1}(0)$ can be expressed trough the Bernoulli numbers

$$E_{2n-1}(0) = -\frac{2}{2n}\,(2^{2n}-1) B_{2n}, \tag{45}$$

and combining (31), (33), (44) and (45), we finally reproduce the celebrated formula

$$\zeta(2n) = (-1)^{n+1} \, \frac{2^{2n-1}}{(2n)!} \, \pi^{2n} \, B_{2n}. \tag{46}$$

5 The general case of $\zeta(2n+1)$

The evaluation of $\zeta(3)$ can be also generalized straightforwardly. We have

$$\zeta(n) = \int\limits_0^1 \frac{Li_{n-1}(x_1)}{x_1} dx_1 = \int\limits_0^1 \frac{dx_1}{x_1} \int\limits_0^{x_1} \frac{Li_{n-2}(x_2)}{x_2}\, dx_2 = \iint\limits_D \frac{Li_{n-2}(x_2)}{x_1 x_2} dx_1 dx_2.$$

Interchanging the order of integrations in the two-dimensional integral, we get

$$\zeta(n) = \int\limits_0^1 \frac{Li_{n-2}(x_2)}{x_2}\, dx_2 \int\limits_{x_2}^1 \frac{dx_1}{x_1} = -\int\limits_0^1 \frac{\ln{(x_2)} Li_{n-2}(x_2)}{x_2}\, dx_2. \tag{47}$$

Now we can repeatedly apply the recurrence relation (24), accompanied with $Li_1(x) = -\ln{(1-x)}$ at the last step, and transform (47) into

$$\zeta(n) = \int\limits_0^1 \frac{\ln x_1}{x_1}\, dx_1 \int\limits_0^{x_1} \frac{dx_2}{x_2} \cdots \int\limits_0^{x_{n-4}} \frac{dx_{n-3}}{x_{n-3}} \int\limits_0^{x_{n-3}} \frac{\ln{(1-x_{n-2})}}{x_{n-2}}\, dx_{n-2},$$

which after rescaling

$$x_2 = x_1 y_2, \quad x_3 = x_2 y_3 = x_1 y_2 y_3, \ldots, \quad x_{n-2} = x_{n-3} y_{n-2} = x_1 y_2 \cdots y_{n-2},$$

takes the form

$$\zeta(n) = \int\limits_0^1 \frac{\ln x_1}{x_1}\, dx_1 \int\limits_0^1 \frac{dy_2}{y_2} \cdots \int\limits_0^1 \frac{dy_{n-3}}{y_{n-3}} \int\limits_0^1 \frac{\ln{(1 - x_1 y_2 \cdots y_{n-2})}}{y_{n-2}}\, dy_{n-2}. \tag{48}$$

Then the relation

$$\int\limits_0^1 \frac{dy_{n-1}}{1 - x_1 y_2 \cdots y_{n-1}} = -\frac{\ln{(1 - x_1 y_2 \cdots y_{n-2})}}{y_1 y_2 \cdots y_{n-2}}$$

shows that (48) is equivalent to the $(n-1)$-dimensional integral

$$\zeta(n) = -\int\limits_{\Box_{n-1}}\cdots\int \frac{\ln x_1}{1 - x_1 \cdots x_{n-1}}\, dx_1 \cdots dx_{n-1}. \tag{49}$$

As in the previous case, (49) can be further transformed into

$$\zeta(n) = -\frac{2^n}{2^n-1} \underset{\square_{n-1}}{\int\cdots\int} \frac{\ln x_1}{1-x_1^2\cdots x_{n-1}^2}\, dx_1\cdots dx_{n-1},$$

or, in the more symmetrical way,

$$\zeta(n) = -\frac{2^n}{2^n-1}\,\frac{1}{n-1} \underset{\square_{n-1}}{\int\cdots\int} \frac{\ln(x_1\cdots x_{n-1})}{1-x_1^2\cdots x_{n-1}^2}\, dx_1\cdots dx_{n-1}. \tag{50}$$

Let us now assume that n is odd and apply the Beukers-Kolk-Calabi change of variables (28) to the integral (50). As the result, we get

$$\zeta(2n+1) = -\frac{1}{2n}\,\frac{2^{2n+1}}{2^{2n+1}-1} \underset{\Delta_{2n}}{\int\cdots\int} \ln\left[\tan(u_1)\cdots\tan(u_{2n})\right] du_1\cdots du_{2n},$$

which is the same as

$$\zeta(2n+1) = -\frac{2^{2n+1}}{2^{2n+1}-1} \underset{\Delta_{2n}}{\int\cdots\int} \ln\left[\tan(u_1)\right] du_1\cdots du_{2n}.$$

By rescaling variables, we can go from the polytope Δ_{2n} to the polytope δ_{2n} in this $2n$-dimensional integral and get

$$\zeta(2n+1) = -\frac{2^{2n+1}}{2^{2n+1}-1}\left(\frac{\pi}{2}\right)^{2n} \underset{\delta_{2n}}{\int\cdots\int} \ln\left[\tan\left(u_1\frac{\pi}{2}\right)\right] du_1\cdots du_{2n}. \tag{51}$$

Using the kernel $K_{2n}(u,v)$, we can reduce the evaluation of (51) to the evaluation of the following one-dimensional integral

$$\zeta(2n+1) = -\frac{2\pi^{2n}}{2^{2n+1}-1}\int\limits_0^1 \ln\left[\tan\left(\frac{\pi}{2}u\right)\right] K_{2n}(u,u)\, du. \tag{52}$$

But

$$\ln\left[\tan\left(\frac{\pi}{2}(1-u)\right)\right] = \ln\left[\cot\left(\frac{\pi}{2}u\right)\right] = -\ln\left[\tan\left(\frac{\pi}{2}u\right)\right],$$

which enables to rewrite (52) as

$$\zeta(2n+1) = -\frac{\pi^{2n}}{2^{2n+1}-1} \int\limits_0^1 \ln\left[\tan\left(\frac{\pi}{2}u\right)\right] \left[K_{2n}(u,u) - K_{2n}(1-u,1-u)\right] du. \tag{53}$$

However, from (42) and (43) we have (recall that $E_{2n-1}(1-u) = -E_{2n-1}(u)$)

$$K_{2n}(u,u) - K_{2n}(1-u,1-u) = (-1)^n \frac{2^{2n-1}}{(2n)!} \frac{d}{du} E_{2n}(u),$$

and the straightforward integration by parts in (53) yields finally the result

$$\zeta(2n+1) = \frac{(-1)^n \, \pi^{2n+1}}{4\left[1 - 2^{-(2n+1)}\right](2n)!} \int\limits_0^1 \frac{E_{2n}(u)}{\sin(\pi u)} \, du. \tag{54}$$

This is exactly the integral representation for $\zeta(2n+1)$ found in [10]. Our earlier result (22) for $\zeta(3)$ is just a special case of this more general formula.

6 concluding remarks: $\zeta(2)$ and amoebas

It remains to clarify the origin of the Beukers-Kolk-Calabi's highly non-trivial miraculous change of variables (28). Maybe an interesting observation due to Passare [7] that $\zeta(2)$ is related to the amoeba of the polynomial $1 - z_1 - z_2$ gives a clue.

Amoebas are fascinating objects in complex geometry [18, 19]. They are defined as follows [20]. For a Laurent polynomial $P(z_1, \ldots, z_n)$, let Z_P denote the zero locus of $P(z_1, \ldots, z_n)$ in $(\mathbb{C}\backslash\{0\})^n$ defined by $P(z_1, \ldots, z_n) = 0$. The amoeba $\mathcal{A}(P)$ of the Laurent polynomial $P(z_1, \ldots, z_n)$ is the image of the complex hypersurface Z_P under the map

$$\mathrm{Log}: \ (\mathbb{C}\backslash\{0\})^n \to \mathbb{R}^n$$

defined through

$$(z_1, \ldots, z_n) \to (\ln|z_1|, \ldots, \ln|z_n|).$$

Let us find the amoeba of the following Laurent polynomial

$$P(z_1, z_2) = z_1 - z_1^{-1} - i\left(z_2 - z_2^{-1}\right). \tag{55}$$

Taking

$$z_1 = e^u\, e^{i\phi_u}, \quad z_2 = e^v\, e^{-i\phi_v},$$

we find that the zero locus of the polynomial (55) is determined by conditions

$$\cos\phi_u\,\sinh u = \sin\phi_v\,\cosh v, \;\; \sin\phi_u\,\cosh u = \cos\phi_v\,\sinh v.$$

If we rewrite these conditions as follows

$$x = \frac{\sinh v}{\cosh u} = \frac{\sin\phi_u}{\cos\phi_v}, \quad y = \frac{\sinh u}{\cosh v} = \frac{\sin\phi_v}{\cos\phi_u}, \tag{56}$$

we immediately recognize the Beukers-Kolk-Calabi substitution (8) and its hyperbolic version with the only difference that in (8) we had $0 \le x, y \le 1$. However, from (56) we get

$$\cos^2\phi_u = \frac{1-x^2}{1-x^2y^2}, \quad \cos^2\phi_v = \frac{1-y^2}{1-x^2y^2}, \tag{57}$$

and

$$\cosh^2 u = \frac{1+y^2}{1-x^2y^2}, \quad \cosh^2 v = \frac{1+x^2}{1-x^2y^2}. \tag{58}$$

It is clear from (57) and (58) that we should have

$$x^2 \le 1, \quad y^2 \le 1.$$

Therefore, the amoeba $\mathcal{A}(P)$ is given by relations

$$\mathcal{A}(P) = \left\{(u,v):\; -1 \le \frac{\sinh u}{\cosh v} \le 1, \;\; -1 \le \frac{\sinh v}{\cosh u} \le 1\right\}, \tag{59}$$

and the hyperbolic version of the Beukers-Kolk-Calabi change of variables (8) transforms the unite square $\Box$ into the one-quarter of the amoeba (59). Then the analog of (9) indicates that $\zeta(2)$ equals to the one-third of the area of this amoeba.

As we see, the hyperbolic version of the Beukers-Kolk-Calabi change of variables seems more fundamental and arises quite naturally in the context of the amoeba (59). Trigonometric version of it then is just an area-preserving transition from the "radial" coordinates (u,v) to the "angular" ones (ϕ_u, ϕ_v).

One more amoeba related to $\zeta(2)$ was found in [7]. Although the corresponding amoeba $\mathcal{A}(1-z_1-z_2)$ looks different from the amoeba (59), they

do have the same area. The trigonometric change of variables used by Passare in [7] is also different from (8) but also leads to simple calculation of the area of $\mathcal{A}(1 - z_1 - z_2)$ and hence $\zeta(2)$. Of course it will be very interesting to generalize this mysterious relations between $\zeta(n)$ and amoebas for $n > 2$ and finally disentangle the mystery. I'm afraid, however, that this game is already not for kids under fifteen.

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